

NO FEEDBACK CARD GUESSING FOR TOP TO RANDOM SHUFFLES

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ABSTRACT. Consider n cards that are labeled 1 through n with n an even integer. The cards are put face down and their ordering starts with card labeled 1 on top through card labeled n at the bottom. The cards are top to random shuffled m times and placed face down on the table. Starting from the top the cards are guessed without feedback (i.e. whether the guess was correct or false and what the guessed card was) one at a time. For $m > 4n \log n + cn$ we find a guessing strategy that maximizes the expected number of correct guesses.

1 Introduction

Assuming that we have n cards, initially ordered 1 at top through n at bottom, applying one *top to random shuffle* to this deck means taking the top card and placing it back into the deck at position i with probability w_i . Diaconis et al. (1992) found that, when $w_i = \frac{1}{n}$ for all i , the eigenvalues of the transition matrix that is associated with this Markov chain are $0, \frac{1}{n}, \dots, \frac{n-2}{n}, 1$ and the multiplicities of the eigenvalues that are of the form $\frac{i}{n}$ are the number of permutations with exactly i fixed points.

A *random to top shuffle*, often referred to as *Tsetlin library scheme* or *move to front scheme*, is the inverse of a top to random shuffle. Considering that we have n records, the move to front scheme is one of the most famous self-organizing rules that has been considered for searching a particular record with minimum cost. McCabe (1965) proved the existence of the limiting average number of records to be examined for a request of information. Phatarfod studied the random to top shuffles on a deck of n cards where the i^{th} card is selected with probability w_i . He determined that it is a Markov chain, but not a random walk on a group and that the stationary distribution of this Markov chain is not uniform. Phatarfod (1994) also computed the m^{th} step transition matrix for this Markov chain by exploiting a connection with the coupon collector's problem. Phatarfod (1991) proved that the non-zero eigenvalues of the transition matrix for the random to top shuffle are real, nonnegative, of the form $w_i, w_i + w_j, \dots, \sum_{i=1}^n w_i$, and the multiplicity of the eigenvalue $\sum w_i$ with sum over m terms is the number of fixed point free permutations of $N - m$ elements. Independently, Fill (1996) derived a formula for the m^{th} step transition matrix. Moreover, he measured the distance of the distribution of a random to top shuffle deck from its stationary distribution using the separation distance. Donnelly (1991) and Kapoor and Reingold (1991) also determined these eigenvalues independently.

¹This research was partially supported by NSF grant DMS 0802082

For top to random shuffles the spectral decomposition of the transition matrix has some algebraic interpretations. Solomon (1976) proved that $A_S = \sum_{D(\pi)=S} \pi$ forms an algebra, where $D(\pi)$ is the descent set of π . Diaconis et al. (1992) showed that, when $S = \{1\}$, $A_{\{1\}}$ generates an n -dimensional commutative semisimple subalgebra of the group algebra, $\mathcal{A}(S_n)$ and $A_{\{1\}}$ has the same spectral decomposition as the transition matrix for top to random shuffles. Garsia and Wallach (2007) also proved the same result. Aldous and Diaconis (1986), Diaconis et al. (1992) and recently Stark (2002) proved that $n \log n + cn$ top to random shuffles is sufficient for the deck to be random.

In this paper we look at the following problem. Consider an ordered deck of n cards labeled 1 through n , where n is an even integer. After m top to random shuffles (we consider $w_i = \frac{1}{n}$ for all i), a person is asked to guess the cards starting with the top card. During the process the guesser receives no information, i.e., neither the type of the card is revealed nor if the guessed card was correct or not. We determine the best no feedback guessing strategy, i.e., the strategy that maximizes the expected number of correct guesses. We prove that, given a deck of n cards, for n even, $c \geq 0$ and $m > 4n \log n + cn$, the best no feedback guessing strategy after m top to random shuffles is to guess card $n - 1$ for positions 1 through $n/2$ and card n for positions $n/2 + 1$ through n . By using this best no feedback card guessing strategy we show that after $m > 4n \log n + cn$ top to random shuffles the deck is close to randomly distributed deck. Ciucu (1998) studied no feedback card guessing for riffle shuffles.

2 Calculation of the m step position matrix

We start by considering the structure of the position matrix for top to random shuffles. The position matrix P is an $n \times n$ matrix whose $(j, k)^{th}$ entry is the probability that card j moves to position k after one shuffle.

Lemma 2.1. *For $1 \leq j, k \leq n$, the position matrix has the form*

$$P_{jk} = \begin{cases} \frac{1}{n} & \text{if } j = 1 \quad 1 \leq k \leq n, \\ \frac{n-(j-1)}{n} & \text{if } j \neq 1 \quad k = j - 1, \\ \frac{j-1}{n} & \text{if } j \neq 1 \quad k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. After one top to random shuffle card 1 (the top card) can go to any position with equal probability $\frac{1}{n}$. If card 1 moves to any position below the $(j-1)^{th}$ position, card j moves to the $(j-1)^{th}$ position. Since there are $n - (j-1)$ such positions, this implies that card j moves to position $j - 1$ with probability $\frac{n-(j-1)}{n}$. Otherwise, if card 1 moves to any position above the position of the j^{th} card, card j stays in the same position. This occurs with probability $\frac{j-1}{n}$. Since these are the only possible positions for card j after one shuffle, all other positions have zero probability. \square

Proposition 2.2. Let $v_k(j)$ represent the j^{th} component of the k^{th} eigenvector of the position matrix P and λ_k the corresponding eigenvalue for v_k . Then

$$v_k(j) = \begin{cases} (-1)^{n+j} \binom{n-k}{j-k} & j \geq k & 1 \leq k \leq n-1, \\ 0 & j < k & 1 \leq k \leq n-1, \\ 1 & 1 \leq j \leq n & k = n, \end{cases}$$

and $\lambda_k = \frac{k-1}{n}$ for all $1 \leq k \leq n-1$ and $\lambda_n = 1$.

Proof. Assume that $j \geq k$ and $1 \leq k \leq n-1$. We have

$$(Pv_k)_{jk} = \frac{n-j+1}{n} v_k(j-1) + \frac{j-1}{n} v_k(j) = (-1)^{n+j} \frac{k-1}{n} \binom{n-k}{j-k} = \lambda_k v_k(j).$$

The other cases are trivial. \square

Let E be the $n \times n$ matrix whose k^{th} column is v_k for $1 \leq k \leq n$. By Proposition 2.2 it follows that $E^{-1}PE$ is a diagonal matrix D , where the k^{th} diagonal element is λ_k , the eigenvalue corresponding to v_k . To compute the probability of card j being in position k after a certain number of shuffles we need the powers of the position matrix P .

Lemma 2.3. The inverse of E is given by

$$E_{jk}^{-1} = \begin{cases} v_k(j) - \frac{1}{n} \sum_{i=1}^j v_i(j) & 1 \leq j < n & 1 \leq k \leq j, \\ -\frac{1}{n} \sum_{i=1}^j v_i(j) & 1 \leq j < n & j < k \leq n, \\ \frac{1}{n} & j = n & 1 \leq k \leq n. \end{cases}$$

Proof. It is enough to show $E^{-1}E = I$. Let $j = k$. We have

$$\begin{aligned} (E^{-1}E)_{jj} &= \sum_{i=1}^j (v_i(j) - \frac{1}{n} \sum_{k=1}^j v_k(j)) v_j(i) + \sum_{i=j+1}^n \left(-\frac{1}{n} \sum_{k=1}^j v_k(j) \right) v_j(i) \\ &= \sum_{i=1}^j v_i(j) v_j(i) - \frac{1}{n} \sum_{k=1}^j v_k(j) \sum_{i=1}^n v_j(i) = v_j(j) v_j(j) = 1. \end{aligned}$$

Similar steps show $(E^{-1}E)_{jk} = 0$ for $j < k$ and $j > k$. \square

Corollary 2.4. The entries of P^m are

$$P_{jk}^m = \begin{cases} \frac{1}{n^{m+1}} \sum_{i=2}^j (-1)^{j+i+1} (i-1)^m \binom{n-i}{j-i} \binom{n}{i-1} + \frac{1}{n} & 1 \leq j < k \leq n, \\ \frac{1}{n^{m+1}} \sum_{i=2}^j (-1)^{j+i+1} (i-1)^m \binom{n-i}{j-i} \binom{n}{i-1} \\ + \frac{1}{n^m} \sum_{i=k}^j (-1)^{j+i} (i-1)^m \binom{n-i}{j-i} \binom{n-k}{i-k} + \frac{1}{n} & 1 \leq k \leq j \leq n. \end{cases}$$

3 The Best Guessing Strategy for Top to Random Shuffles

We now determine the best guessing strategy for top to random shuffles by using the structure of the m^{th} power of the transition matrix P derived in Corollary 2.4. The final result is given by Theorem 3.13.

Proposition 3.1. *Let $n \geq 4$ be even and $c \geq 0$. For $m > n \log n + cn$, P_{nk}^m is increasing in k for $1 \leq k \leq n$.*

Proof. Consider

$$P_{n(k+1)}^m - P_{nk}^m = \sum_{i=k}^{n-1} (-1)^{n+i+1} (i-1)^m \frac{n-i}{n-k} \binom{n-k}{i-k},$$

and define $a_i := (i-1)^m \frac{n-i}{n-k} \binom{n-k}{i-k}$. For $k \leq i \leq n-2$, the quotient satisfies

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{i-1}\right)^m \frac{n-i-1}{i+1-k} > \left(1 + \frac{1}{n-3}\right)^m \frac{1}{n-1} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n}.$$

The first inequality holds, because $\frac{n-i-1}{i+1-k} > \frac{n-i-1}{i+1} > \frac{1}{n-1}$.

Let $c \geq 0$. We know that $m \log(1 + \frac{1}{n-1}) = m \sum_{t=1}^{\infty} \frac{1}{t} (\frac{1}{n})^t$. Hence, for $m > n \log n + cn$, $m \log(1 + \frac{1}{n-1}) > \log n + c$. This implies $\frac{a_{i+1}}{a_i} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n} > e^c$. Therefore, a_i is increasing for $m > n \log n + cn$ and $k \leq i \leq n-1$. Hence, for all $m > n \log n + cn$, $P_{n(k+1)}^m - P_{nk}^m = \sum_{i=k}^{n-1} (-1)^{i+1} a_i > 0$. \square

Proposition 3.2. *Let $n \geq 4$ be even and $c \geq 0$. For $m > n \log 2n + cn$ and $1 \leq j \leq n-1$ fixed, P_{jk}^m is decreasing in k for $1 \leq k \leq n$.*

Proof. Case 1: $k = j$

$$P_{jj}^m - P_{j(j+1)}^m = \frac{(j-1)^m}{n^m} \geq 0 \text{ for all } m.$$

Case 2: $k > j$

$$P_{jk}^m - P_{j(k+1)}^m = 0 \text{ trivially for all } m.$$

Case 3: $k < j$

$$P_{jk}^m - P_{j(k+1)}^m = \frac{1}{n^m} \binom{n-k}{n-j} (-1)^{j+k} \sum_{i=0}^{j-k} (-1)^i (i+k-1)^m \binom{j-k}{i} \left(1 - \frac{i}{n-k}\right).$$

Define

$$a_i := (i+k-1)^m \binom{j-k}{i} \left(1 - \frac{i}{n-k}\right) \quad (3.1)$$

and let $c \geq 0$. We observe below that a_i , defined as in (3.1), is increasing for $m > n \log 2n + cn$. For $0 \leq i \leq j-k-1$,

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{i+k-1}\right)^m \frac{n-k-i-1}{n-k-i} \frac{j-k-i}{i+1}$$

$$\geq \left(1 + \frac{1}{n-3}\right)^m \frac{1}{2(n-2)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{2n}.$$

The first inequality holds, since $i \leq j - k - 1$, $j \leq n - 1$, and $k \geq 1$. For $m > n \log 2n + cn$, we get $m \log(1 + \frac{1}{n-1}) = m \sum_{t=1}^{\infty} \frac{1}{t} (\frac{1}{n})^t > \frac{m}{n} > \log 2n + c$. Hence, $\frac{a_{i+1}}{a_i} \geq (1 + \frac{1}{n-1})^m \frac{1}{2n} > e^c$ and so a_i is increasing. We conclude that for $m > n \log 2n + cn$, $\sum_{i=0}^{j-k} (-1)^{i+j+k} a_i > 0$ and so $P_{jk}^m - P_{j(k+1)}^m > 0$. \square

In the next lemma we state some properties of the binomials used in Proposition 3.4, Proposition 3.5 and Corollary 3.6.

Lemma 3.3. *Let $n \geq 10$ be even, then*

- (i) $\frac{1}{n} \binom{n}{i-1} - \binom{n/2-1}{i-1-n/2} > 0$ for all $\frac{n}{2} + 1 \leq i \leq n - 1$.
- (ii) $\frac{\frac{1}{n} \binom{n}{i} - \binom{n/2-1}{i-n/2}}{\frac{1}{n} \binom{n}{i-1} - \binom{n/2-1}{i-1-n/2}}$ is decreasing in i for $\frac{n}{2} + 1 \leq i \leq n - 2$.
- (iii) $\frac{\frac{1}{n} \binom{n}{i} - \binom{n/2-1}{i-n/2}}{\frac{1}{n} \binom{n}{i-2} - \binom{n/2-1}{i-2-n/2}}$ is decreasing in i for all $\frac{n}{2} + 2 \leq i \leq n - 2$.
- (iv) $\frac{\binom{n-i-1}{j-i-1}-1}{\binom{n-i}{j-i}-1}$ is decreasing in i for $1 \leq i \leq j - 2$ and $3 \leq j \leq n - 1$.

Proposition 3.4. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 3n \log n + cn$, we have $P_{n(\frac{n}{2}+1)}^m > P_{j(\frac{n}{2}+1)}^m$ for $1 \leq j \leq \frac{n}{2}$.*

Proof. We have

$$\begin{aligned} P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m &= \frac{1}{n^m} \left(\sum_{i=2}^j (-1)^i (i-1)^m \frac{1}{n} \binom{n}{i-1} \left((-1)^{j+2} \binom{n-i}{j-i} - 1 \right) \right. \\ &\quad + \sum_{i=j+1}^{n/2} (-1)^{n+i+1} (i-1)^m \frac{1}{n} \binom{n}{i-1} \\ &\quad \left. + \sum_{i=n/2+1}^{n-1} (-1)^{i+1} (i-1)^m \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2-1}{i-n/2-1} \right) \right). \end{aligned}$$

Case 1: If j is even, the equality above simplifies to

$$P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m = \frac{1}{n^m} \sum_{i=2}^{n-2} (-1)^i a_i,$$

where a_i is defined as

$$a_i := \begin{cases} (i-1)^m \frac{1}{n} \binom{n}{i-1} \left(\binom{n-i}{j-i} - 1 \right) & 2 \leq i \leq j-1, \\ i^m \frac{1}{n} \binom{n}{i} & j \leq i \leq \frac{n}{2} - 1, \\ i^m \left(\frac{1}{n} \binom{n}{i} - \binom{n/2-1}{i-n/2} \right) & \frac{n}{2} \leq i \leq n-2. \end{cases} \quad (3.2)$$

Case 2: If j is odd, the equality above becomes

$$P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m = \frac{1}{n^m} \sum_{i=1}^{n-2} (-1)^i a_i,$$

where a_i is defined as

$$a_i = \begin{cases} i^m \frac{1}{n} \binom{n}{i} \left(\binom{n-i-1}{j-i-1} + 1 \right) & 1 \leq i \leq j-1, \\ i^m \frac{1}{n} \binom{n}{i} & j \leq i \leq \frac{n}{2} - 1, \\ i^m \left(\frac{1}{n} \binom{n}{i} - \binom{n/2-1}{i-n/2} \right) & \frac{n}{2} \leq i \leq n-2. \end{cases} \quad (3.3)$$

Using equation (3.2) we have the following bounds for $\frac{a_{i+1}}{a_i}$.

Case 1.a: For $2 \leq i \leq j-2$,

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1} \right)^m \frac{n-i+1}{i} \frac{\binom{n-i-1}{j-i-1} - 1}{\binom{n-i}{j-i} - 1} \\ &\geq \left(1 + \frac{2}{n-8} \right)^m \frac{n+8}{n-6} \frac{2(n+2)}{(n+4)(n+1)} > \left(1 + \frac{2}{n-8} \right)^m \frac{2}{n-6} \\ &> \left(1 + \frac{1}{n-1} \right)^m \frac{1}{n^3}. \end{aligned}$$

The first inequality holds, since $j \leq \frac{n}{2} - 1$ and by Lemma 3.3 part (iv) $\frac{\binom{n-i-1}{j-i-1} - 1}{\binom{n-i}{j-i} - 1} >$

$$\frac{\binom{n-j+1}{1} - 1}{\binom{n-j+2}{2} - 1}.$$

Case 1.b: For $i = j-1$,

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{2}{j-2} \right)^m \frac{n-j+2}{j(j-1)} \frac{n-j+1}{n-j} \\ &> \left(1 + \frac{4}{n-6} \right)^m \frac{2}{n-2} > \left(1 + \frac{1}{n-1} \right)^m \frac{1}{n^3}. \end{aligned}$$

The first inequality follows, since $j \leq \frac{n}{2} - 1$.

Case 1.c: For $j \leq i \leq \frac{n}{2} - 2$,

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{i} \right)^m \frac{n-i}{i+1} > \left(1 + \frac{2}{n-4} \right)^m > \left(1 + \frac{1}{n-1} \right)^m \frac{1}{n^3}.$$

Case 1.d: For $i = \frac{n}{2} - 1$,

$$\begin{aligned}\frac{a_{i+1}}{a_i} &= \left(1 + \frac{2}{n-2}\right)^m \left(\frac{n+2}{n} - \frac{n}{\binom{n}{n/2-1}}\right) \\ &\geq \left(1 + \frac{2}{n-2}\right)^m \frac{2}{n} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^3}.\end{aligned}$$

The first inequality holds, since for all $n \geq 6$, $\binom{n}{n/2-1} > \binom{n}{1}$.

Case 1.e: For $\frac{n}{2} \leq i \leq n-3$, we have

$$\begin{aligned}\frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i}\right)^m \frac{\frac{1}{n}\binom{n}{i+1} - \binom{n/2-1}{i+1-n/2}}{\frac{1}{n}\binom{n}{i} - \binom{n/2-1}{i-n/2}} \geq \left(1 + \frac{1}{n-3}\right)^m \frac{12}{(n-2)(n+8)} \\ &> \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^3}.\end{aligned}$$

The first inequality holds, by Lemma 3.3 part (ii) $\frac{\frac{1}{n}\binom{n}{i+1} - \binom{n/2-1}{i+1-n/2}}{\frac{1}{n}\binom{n}{i} - \binom{n/2-1}{i-n/2}} \geq \frac{\frac{1}{n}\binom{n}{n-2} - \binom{n/2-1}{n/2-2}}{\frac{1}{n}\binom{n}{n-3} - \binom{n/2-1}{n/2-3}}$.

Using equation (3.3) we have the following bounds for $\frac{a_{i+1}}{a_i}$.

Case 2.a: For $1 \leq i \leq j-2$, we have

$$\begin{aligned}\frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i}\right)^m \frac{n-i}{i+1} \frac{1 + \binom{n-i-2}{j-i-2}}{1 + \binom{n-i-1}{j-i-1}} > \left(1 + \frac{1}{i}\right)^m \frac{n-i}{n-i-1} \frac{j-i-1}{2(i+1)} \\ &\geq \left(1 + \frac{2}{n-6}\right)^m \frac{1}{n-4} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^3}.\end{aligned}$$

The first inequality holds, since $j \leq \frac{n}{2} - 1$ and $\frac{1 + \binom{n-i-2}{j-i-2}}{1 + \binom{n-i-1}{j-i-1}} > \frac{\binom{n-i-2}{j-i-2}}{2\binom{n-i-1}{j-i-1}}$. The second inequality also follows, since $j \leq \frac{n}{2} - 1$.

Case 2.b: For $i = j-1$, we have

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{j-1}\right)^m \frac{n-j+1}{2j} > \left(1 + \frac{2}{n-4}\right)^m \frac{1}{2} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^3}.$$

The first inequality follows, since $j \leq \frac{n}{2} - 1$.

Note that the remaining three cases, which are $j \leq i \leq \frac{n}{2} - 2$, $i = \frac{n}{2} - 1$ and $\frac{n}{2} \leq i \leq n-3$, are identical to the cases 1.c, 1.d and 1.e.

Let $c \geq 0$. For $m > 3n \log n + cn$, we get $\frac{m}{n} > 3 \log n + c$ and, since $m \log(1 + \frac{1}{n-1}) = m \sum_{t=1}^{\infty} \frac{1}{t} \left(\frac{1}{n}\right)^t$, this implies $m \log(1 + \frac{1}{n-1}) > 3 \log n + c$. Hence, $(1 + \frac{1}{n-1})^m \frac{1}{n^3} > e^c$. Let us consider the ratio $\frac{a_{i+1}}{a_i}$ for both j even and odd cases. In all cases above we show $\frac{a_{i+1}}{a_i} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^3}$. Therefore, for $2 \leq i \leq n-2$, a_i , defined as in (3.2) and (3.3) is increasing for $m > 3n \log n + cn$. We conclude that for $m > 3n \log n + cn$ and

j odd, $\sum_{i=1}^{n-2} (-1)^i a_i = (-a_1 + a_2) + (-a_3 + a_4) + \dots + (-a_{n-2} + a_{n-1}) > 0$. Similarly, for j even, $\sum_{i=2}^{n-2} (-1)^i a_i = a_2 + \sum_{i=3}^{n-2} (-1)^i a_i > 0$. Hence, $P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m > 0$ for both j even and odd. \square

Proposition 3.5. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 4n \log n + cn$, we have $P_{n(\frac{n}{2}+1)}^m > P_{j(\frac{n}{2}+1)}^m$ for $\frac{n}{2} + 1 \leq j \leq n - 1$.*

Proof. Consider

$$\begin{aligned} P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m &= \frac{1}{n^m} \left(\sum_{i=2}^{n/2} (-1)^i (i-1)^m \binom{n}{i-1} \frac{1}{n} \left((-1)^{j+2} \binom{n-i}{j-i} - 1 \right) \right. \\ &\quad + \sum_{i=n/2+1}^j (-1)^i (i-1)^m \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2-1}{i-1-n/2} \right) \\ &\quad \times \left((-1)^{j+2} \binom{n-i}{j-i} - 1 \right) \\ &\quad \left. + \sum_{i=j+1}^{n-1} (-1)^{i+1} (i-1)^m \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2-1}{i-1-n/2} \right) \right). \end{aligned}$$

The proof for $P_{n(\frac{n}{2}+1)}^m - P_{j(\frac{n}{2}+1)}^m > 0$ for $m > 4n \log n + cn$ and $c \geq 0$ is analog to the proof of Proposition 3.4. \square

Corollary 3.6. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 4n \log n + cn$ and $1 \leq j \leq n - 1$ fixed, we have $P_{nk}^m > P_{jk}^m$ for all $\frac{n}{2} + 1 \leq k \leq n$.*

Proof. This result follows from Proposition 3.1, Proposition 3.2, Proposition 3.4 and Proposition 3.5. \square

The next lemma states some properties of the binomials used in Proposition 3.8, Proposition 3.10, Proposition 3.11 and Corollary 3.9.

Lemma 3.7. *Let $n \geq 8$ be even, then*

- (i) $\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} > 0$ for $\frac{n}{2} \leq i \leq n - 2$.
- (ii) $\frac{\frac{1}{n} \binom{n}{i} - \binom{n/2}{i+1-n/2}}{\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2}}$ is decreasing in i for $\frac{n}{2} \leq i \leq n - 3$.
- (iii) $\frac{\frac{1}{n} \binom{n}{i} - \binom{n/2}{i+1-n/2}}{\frac{1}{n} \binom{n}{i-2} - \binom{n/2}{i-1-n/2}}$ is decreasing in i for $\frac{n}{2} + 1 \leq i \leq n - 3$.

(iv) $\frac{\binom{n-i-1}{j-i-1} - \binom{n-i-1}{j-i}}{\binom{n-i}{j-i} - \binom{n-i}{j-i-1}}$ is decreasing in i for $2 \leq i \leq j-3$ and $5 \leq j \leq n-2$.

Proposition 3.8. Let $n \geq 8$ be even and $c \geq 0$. For $m > n \log n + cn$, we have $P_{(n-1)\frac{n}{2}}^m > P_{n\frac{n}{2}}^m$.

Proof.

$$\begin{aligned} P_{(n-1)\frac{n}{2}}^m - P_{n\frac{n}{2}}^m &= \frac{1}{n^m} \left(\sum_{i=2}^{n/2-1} (-1)^i (i-1)^m \frac{n-i+1}{n} \binom{n}{i-1} \right. \\ &\quad \left. + \sum_{i=n/2}^{n-2} (-1)^i (i-1)^m (n-i+1) \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} \right) + (n-2)^m \right) \\ &= \frac{1}{n^m} \left(\sum_{i=2}^{n-2} (-1)^i a_i + (n-2)^m \right), \end{aligned}$$

where a_i is defined as

$$a_i = \begin{cases} (i-1)^m \frac{n-i+1}{n} \binom{n}{i-1} & 2 \leq i \leq \frac{n}{2} - 1, \\ (i-1)^m (n-i+1) \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} \right) & \frac{n}{2} \leq i \leq n-2. \end{cases} \quad (3.4)$$

Using equation (3.4) we get the following bounds for $\frac{a_{i+1}}{a_i}$.

Case 1.a: For $2 \leq i \leq \frac{n}{2} - 2$, we have

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{i-1} \right)^m \frac{n}{i} - 1 > \left(1 + \frac{2}{n-6} \right)^m > \left(1 + \frac{1}{n-1} \right)^m \frac{1}{n}.$$

Case 1.b: For $i = \frac{n}{2} - 1$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{2}{n-4} \right)^m \frac{n+2}{n+4} \frac{\frac{1}{n} \binom{n}{n/2-1} - 1}{\frac{1}{n} \binom{n}{n/2-2}} > \left(1 + \frac{2}{n-4} \right)^m \frac{1}{n} \\ &> \left(1 + \frac{1}{n-1} \right)^m \frac{1}{n}. \end{aligned}$$

The first inequality follows from $\frac{\frac{1}{n} \binom{n}{n/2-1} - 1}{\frac{1}{n} \binom{n}{n/2-2}} > \frac{\binom{n}{n/2-1}}{\binom{n}{n/2-2}} - \frac{n}{\binom{n}{1}} = \frac{6}{(n-2)}$ for all $n \geq 6$.

Case 1.c: For $\frac{n}{2} \leq i \leq n-3$, we have

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{i-1} \right)^m \frac{n-i}{n-i+1} \frac{\frac{1}{n} \binom{n}{i} - \binom{n/2}{i+1-n/2}}{\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2}}$$

$$> \left(1 + \frac{1}{n-4}\right)^m \frac{3(n-4)}{2(n^2-4n+6)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n}.$$

The first inequality holds by Lemma 3.7 part (ii), $\frac{\frac{1}{n}\binom{n}{i}-\binom{n/2}{i+1-n/2}}{\frac{1}{n}\binom{n}{i-1}-\binom{n/2}{i-n/2}} \geq \frac{\frac{1}{n}\binom{n}{n-3}-\binom{n/2}{n/2-2}}{\frac{1}{n}\binom{n}{n-4}-\binom{n/2}{n/2-3}}.$

The second inequality holds for all $n \geq 6$.

Let $c \geq 0$. For $m > n \log n + cn$, we get $\frac{m}{n} > \log n + c$ and, since $m \log(1 + \frac{1}{n-1}) = m \sum_{t=1}^{\infty} \frac{1}{t} (\frac{1}{n})^t$ this implies $m \log(1 + \frac{1}{n-1}) > \log n + c$. Hence, $(1 + \frac{1}{n-1})^m \frac{1}{n} > e^c$. Above we prove that $\frac{a_{i+1}}{a_i} > (1 + \frac{1}{n-1})^m \frac{1}{n}$. Therefore, for $2 \leq i \leq n-2$, a_i , defined as in (3.4) is increasing and so, $P_{(n-1)\frac{n}{2}}^m - P_{\frac{n}{2}}^m > 0$ for all $m > n \log n + cn$. \square

Corollary 3.9. *Let $n \geq 8$ be even and $c \geq 0$. For $m > n \log 2n + cn$, we have $P_{(n-1)k}^m > P_{nk}^m$ for $1 \leq k \leq \frac{n}{2}$.*

Proof. This result follows from Proposition 3.1, Proposition 3.2 and Proposition 3.8. \square

Proposition 3.10. *Let $n \geq 8$ be even and $c \geq 0$. For $m > 2n \log n + cn$ and $1 \leq j \leq n-2$ fixed, $P_{(n-1)k}^m - P_{jk}^m$ is decreasing in k for $1 \leq k \leq \frac{n}{2}$.*

Proof. Case 1: For $j > k$, we have

$$\begin{aligned} & P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m \\ &= (-1)^{k-1} \left(\frac{k-1}{n}\right)^m (n-k) - (-1)^{j+k} \left(\frac{k-1}{n}\right)^m \binom{n-k}{j-k} \\ &+ \frac{1}{n^m} \left(\sum_{i=k+1}^j (i-1)^m \binom{n-k-1}{i-k} \right) \\ &\times \left((-1)^{j+i+1} \binom{n-i}{j-i} + (-1)^{n-1+i} (n-i) \right) \\ &+ \sum_{i=j+1}^{n-1} (-1)^{n-1+i} (i-1)^m (n-i) \binom{n-k-1}{i-k} \Bigg). \end{aligned}$$

Case 1.1: For j even, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \frac{1}{n^m} \sum_{i=k}^{n-1} (-1)^{i-1} a_i,$$

where a_i is defined as

$$a_i = \begin{cases} (i-1)^m \left(\binom{n-i}{j-i} + n-i \right) \binom{n-k-1}{i-k} & k \leq i \leq j, \\ (i-1)^m (n-i) \binom{n-k-1}{i-k} & j+1 \leq i \leq n-1. \end{cases} \quad (3.5)$$

Case 1.2: For j odd, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \frac{1}{n^m} \sum_{i=k}^{n-2} (-1)^i a_i,$$

where a_i is defined as

$$a_i = \begin{cases} (i-1)^m \left(\binom{n-i}{j-i} - n + i \right) \binom{n-k-1}{i-k} & k \leq i \leq j-2, \\ i^m (n-i-2) \binom{n-k-1}{i+1-k} & i = j-1, \\ i^m (n-i-1) \binom{n-k-1}{i+1-k} & j \leq i \leq n-2. \end{cases} \quad (3.6)$$

Case 2: For $j = k$, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \frac{1}{n^m} \left(((-1)^{n-1+k} (n-k) - 1) (k-1)^m + \sum_{i=k+1}^{n-1} (-1)^{n-1+i} (i-1)^m (n-i) \binom{n-k-1}{i-k} \right).$$

Case 2.1: For j even, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \frac{1}{n^m} \sum_{i=k}^{n-1} (-1)^{i-1} a_i,$$

where a_i is defined as

$$a_i = \begin{cases} (i-1)^m (n-i+1) & i = k, \\ (i-1)^m (n-i) \binom{n-k-1}{i-k} & k+1 \leq i \leq n-1. \end{cases} \quad (3.7)$$

Case 2.2: For j odd, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \frac{1}{n^m} \sum_{i=k}^{n-1} (-1)^{i-1} a_i,$$

where a_i is defined as

$$a_i = \begin{cases} (i-1)^m (n-i-1) & i = k, \\ (i-1)^m (n-i) \binom{n-k-1}{i-k} & k+1 \leq i \leq n-1. \end{cases} \quad (3.8)$$

Case 3: For $j < k$, we have

$$P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m$$

$$= \frac{1}{n^m} \sum_{i=k}^{n-1} (-1)^{i-1} (i-1)^m (n-i) \binom{n-k-1}{i-k} = \frac{1}{n^m} \sum_{i=k}^{n-1} (-1)^{i-1} a_i,$$

where a_i is defined as

$$a_i = (i-1)^m (n-i) \binom{n-k-1}{i-k}. \quad (3.9)$$

Using equation (3.5) we get the following cases for $\frac{a_{i+1}}{a_i}$.

Case 1.1.a: For $k \leq i \leq j-1$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1}\right)^m \frac{n-i-1}{i-k+1} \frac{\binom{n-i-1}{j-i-1} + n-i-1}{\binom{n-i}{j-i} + n-i} \\ &> \left(1 + \frac{1}{i-1}\right)^m \frac{(n-i-1)(j-i)}{2(i-k+1)(n-i)} \\ &> \left(1 + \frac{1}{n-4}\right)^m \frac{1}{3(n-3)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

The first inequality holds, since $\frac{\binom{n-i-1}{j-i-1} + n-i-1}{\binom{n-i}{j-i} + n-i} > \frac{\binom{n-i-1}{j-i-1}}{2\binom{n-i}{j-i}} = \frac{j-i}{2(n-i)}$.

The second inequality holds, since $k \leq i \leq j-1$, $i-k+1 \leq j-k$ and, since $j \leq n-2$ and $k \geq 1$, we get $j-k \leq n-3$. Hence, $\frac{j-i}{i-k+1} \geq \frac{1}{n-3}$.

Case 1.1.b: For $i = j$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{j-1}\right)^m \frac{n-j-1}{n-j+1} \frac{n-j-1}{j-k+1} \\ &> \left(1 + \frac{1}{n-3}\right)^m \frac{1}{3(n-2)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

Case 1.1.c: For $j+1 \leq i \leq n-2$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1}\right)^m \frac{n-i-1}{n-i} \frac{n-i-1}{i-k+1} \\ &> \left(1 + \frac{1}{n-3}\right)^m \frac{1}{2(n-2)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

Using equation (3.6) we get the following cases for $\frac{a_{i+1}}{a_i}$.

Case 1.2.a: For $k \leq i \leq j-3$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1}\right)^m \frac{n-i-1}{i-k+1} \frac{\binom{n-i-1}{j-i-1} - (n-i-1)}{\binom{n-i}{j-i} - (n-i)} \\ &> \left(1 + \frac{1}{j-4}\right)^m \frac{n-j+2}{j-k-2} \frac{3(n-j+2)}{(n-j+3)(n-j+4)} \end{aligned}$$

$$\geq \left(1 + \frac{1}{n-6}\right)^m \frac{8}{5(n-5)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}.$$

The first inequality holds, since Lemma 3.7 part (iv) implies,

$$\frac{\binom{n-i-1}{j-i-1} - \binom{n-i-1}{j-i}}{\binom{n-i}{j-i} - \binom{n-i}{j-i-1}} \geq \frac{\binom{n-j+2}{2} - \binom{n-j+2}{1}}{\binom{n-i}{3} - \binom{n-i}{2}}. \text{ The second inequality holds, since } j \leq n-2.$$

Case 1.2.b: For $i = j-2$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{2}{j-3}\right)^m \frac{2(n-j+1)(n-j)}{(j-k)(j-k-1)(n-j+2)} \\ &> \left(1 + \frac{2}{n-5}\right)^m \frac{3}{(n-3)(n-4)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

The first inequality holds, since $j \leq n-2$.

Case 1.2.c: For $i = j-1$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{j-1}\right)^m \frac{n-j-1}{j-k+1} \\ &\geq \left(1 + \frac{1}{n-3}\right)^m \frac{1}{n-2} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

The first inequality holds, since $j \leq n-2$ and $k \geq 1$.

Case 1.2.d: For $j \leq i \leq n-3$, the case is identical to case 1.1.3.

Using equation (3.7) we get the following cases for $\frac{a_{i+1}}{a_i}$.

Case 2.1.a: For $i = k$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{k-1}\right)^m \frac{(n-k-1)^2}{n-k+1} \\ &> \left(1 + \frac{2}{n-2}\right)^m \frac{n^2-4}{2(n+2)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

The first inequality holds, since $k \leq \frac{n}{2}$.

Case 2.1.b: For $k+1 \leq i \leq n-2$, we have

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1}\right)^m \frac{(n-i-1)^2}{(n-i)(i+1-k)} \\ &> \left(1 + \frac{1}{n-2}\right)^m \frac{1}{2} \frac{1}{n-1} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

Using equation (3.8) we get the following cases for $\frac{a_{i+1}}{a_i}$.

Case 2.2.a: For $i = k$, we have

$$\frac{a_{i+1}}{a_i} = \left(1 + \frac{1}{k-1}\right)^m (n-k-1)$$

$$\geq \left(1 + \frac{2}{n-2}\right)^m \frac{n-2}{2} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}.$$

The first inequality holds, since $k \leq \frac{n}{2}$.

Case 2.2.b: For $k+1 \leq i \leq n-2$, the case is identical to case 2.1.b.

Using equation (3.9) we get the following cases for $\frac{a_{i+1}}{a_i}$.

Case 3.1: For $k \leq i \leq n-2$, the cases for j even and odd are the same.

$$\begin{aligned} \frac{a_{i+1}}{a_i} &= \left(1 + \frac{1}{i-1}\right)^m \frac{(n-i-1)^2}{(n-i)(i+1-k)} \\ &> \left(1 + \frac{1}{n-2}\right)^m \frac{1}{2(n-1)} > \left(1 + \frac{1}{n-1}\right)^m \frac{1}{n^2}. \end{aligned}$$

Let $c \geq 0$. For $m > 2n \log n + cn$, we get $\frac{m}{n} > 2 \log n + c$ and since $m \log(1 + \frac{1}{n-1}) = m \sum_{t=1}^{\infty} \frac{1}{t} (\frac{1}{n})^t$ this implies $m \log(1 + \frac{1}{n-1}) > 2 \log n + c$. Hence, $(1 + \frac{1}{n-1})^m \frac{1}{n^2} > e^c$. In all cases above we show that $\frac{a_{i+1}}{a_i} > (1 + \frac{1}{n-1})^m \frac{1}{n^2}$. Therefore, a_i , defined as in (3.5) and (3.6) for $j > k$, (3.7) and (3.8) for $j = k$ and finally (3.9) for $j < k$, is increasing and hence, $P_{(n-1)k}^m - P_{(n-1)(k+1)}^m + P_{j(k+1)}^m - P_{jk}^m = \sum_{i=k}^{n-1} (-1)^{i-1} a_i > 0$, for all $m > 2n \log n + cn$. \square

Proposition 3.11. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 3n \log n + cn$, we have $P_{(n-1)\frac{n}{2}}^m > P_{j\frac{n}{2}}^m$ for $1 \leq j \leq n-2$.*

Proof. Case 1: Let $j < \frac{n}{2}$.

$$\begin{aligned} P_{(n-1)\frac{n}{2}} - P_{j\frac{n}{2}} &= \frac{1}{n^m} \left(\sum_{i=2}^j (-1)^i (i-1)^m \frac{1}{n} \binom{n}{i-1} \left(n-i + (-1)^j \binom{n-i}{j-i} \right) \right. \\ &\quad + \sum_{i=j+1}^{n/2-1} (-1)^i (i-1)^m \frac{n-i}{n} \binom{n}{i-1} \\ &\quad \left. + \sum_{i=n/2}^{n-1} (-1)^i (i-1)^m (n-i) \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} \right) \right). \end{aligned}$$

Case 2: Let $j = \frac{n}{2}$.

$$\begin{aligned} P_{(n-1)\frac{n}{2}}^m - P_{\frac{n}{2}\frac{n}{2}}^m &= \frac{1}{n^m} \left(\sum_{i=2}^{n/2-1} (-1)^{i+1} (i-1)^m \frac{1}{n} \binom{n}{i-1} \left((-1)^{(n/2+1)} \binom{n-i}{n/2-i} - (n-i) \right) \right. \\ &\quad \left. + (-1)^{n/2} \left(\frac{n}{2} - 1 \right)^m \left(\frac{n}{2} + (-1)^{n/2} \right) \left(\frac{1}{n} \binom{n}{n/2-1} - 1 \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=n/2}^{n-3} (-1)^{i+1} i^m (n - (i+1)) \left(\frac{1}{n} \binom{n}{i} - \binom{n/2}{i+1-n/2} \right) + \frac{(n-2)^m}{2} \\
& = \frac{1}{n^m} \left(\sum_{i=2}^{n-3} (-1)^{i+1} a_i + \frac{(n-2)^m}{2} \right).
\end{aligned}$$

Case 3: Let $j > \frac{n}{2}$.

$$\begin{aligned}
& P_{(n-1)\frac{n}{2}}^m - P_{j\frac{n}{2}}^m \\
& = \frac{1}{n^m} \left(\sum_{i=2}^{n/2-1} (-1)^i (i-1)^m \frac{1}{n} \binom{n}{i-1} \left(n-i + (-1)^j \binom{n-i}{j-i} \right) \right. \\
& \quad + \sum_{i=n/2}^j (-1)^i (i-1)^m \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} \right) \left(n-i + (-1)^j \binom{n-i}{j-i} \right) \\
& \quad \left. + \sum_{i=j+1}^{n-1} (-1)^i (i-1)^m (n-i) \left(\frac{1}{n} \binom{n}{i-1} - \binom{n/2}{i-n/2} \right) \right).
\end{aligned}$$

Parallel to the proof of Proposition 3.10 we consider both even and odd subcases to prove that $P_{(n-1)\frac{n}{2}}^m - P_{j\frac{n}{2}}^m > 0$ for $m > 3n \log n + cn$ and $c \geq 0$. \square

Corollary 3.12. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 3n \log n + cn$ and $1 \leq j \leq n-2$ fixed, we have $P_{(n-1)k}^m \geq P_{jk}^m$ for $1 \leq k \leq \frac{n}{2}$.*

Proof. The proof follows from Proposition 3.10 and Proposition 3.11. \square

Theorem 3.13. *Let $n \geq 10$ be even and $c \geq 0$. For $m > 4n \log n + cn$, the best no feedback guessing strategy is to guess card $n-1$ for positions 1 through $n/2$ and card n for positions $n/2+1$ through n .*

Proof. The proof follows from Corollaries 3.6, 3.9, and 3.12. \square

4 Convergence To Uniformity

In this section we use Theorem 3.13 to determine the number of top to random shuffles needed for a deck of cards to become uniformly distributed, i.e., all orderings are equally likely. Here, the distance used to measure the difference between a top to random shuffled deck and a uniform deck is given by the difference of the average number of correct guesses using the best no feedback guessing strategy and that of the uniform distribution. Note that without feedback the expected number of correct guesses for a uniformly distributed deck using any strategy is 1.

Theorem 4.1. *Let $n \geq 10$ be even, $c \geq 0$ and $m > 4n \log n + cn$. Further, let $E^m(n)$ denote the expected number of correct guesses using the best no feedback guessing strategy for a deck of n cards after m top to random shuffles. Then $E^m(n) - 1 \leq e^{-c}$.*

Proof. By Theorem 3.13, $E^m(n) - 1 = \sum_{i=1}^{n/2} P_{(n-1)i}^m + \sum_{i=n/2+1}^n P_{ni}^m - 1$ for all $m > 4n \log n + cn$.

$$\begin{aligned}
& \sum_{i=1}^{n/2} P_{(n-1)i}^m + \sum_{i=n/2+1}^n P_{ni}^m - 1 \leq \frac{n}{2} P_{(n-1)1}^m + \frac{n}{2} P_{nn}^m - 1 \\
& < \frac{n}{2} \left(\frac{1}{n^m} \sum_{i=1}^{n-1} (-1)^{n-1+i} (i-1)^m (n-i) \binom{n-1}{i-1} \right. \\
& \quad \left. + \frac{1}{n^{m+1}} \sum_{i=2}^{n-1} (-1)^{n+1+i} (i-1)^m \binom{n}{i-1} + \frac{2}{n} \right) - 1 \\
& = \frac{1}{2n^{m-1}} ((n-1)! S(m, n-2) + (n-1)! S(m, n) + (n-1)^m - n^{m-1}) \\
& < \frac{1}{2n^{m-1}} \left(\sum_{k=0}^m S(m, k) (n-1)_k + \frac{1}{n} \sum_{k=0}^m S(m, k) (n)_k + (n-1)^m - n^{m-1} \right) \\
& = \frac{1}{2n^{m-1}} \left((n-1)^m + \frac{n^m}{n} + (n-1)^m - n^{m-1} \right) \\
& = n \left(1 - \frac{1}{n} \right)^m < n e^{-\frac{1}{n}m} < n e^{-\frac{1}{n}(4n \log n + cn)} \leq e^{-c}.
\end{aligned}$$

The first inequality holds by Proposition 3.1 and Proposition 3.2. The second inequality holds, since the first term in $P_{(n-1)1}^m$, namely $\sum_{i=2}^{n-1} (-1)^{n+i} (i-1)^m (n-i) \binom{n}{i-1} = n! (S(m, n-1) + S(m, n)) - n^m < 0$, where $S(m, n)$ is the Stirling number of the second kind. Similarly, the second term in P_{nn}^m , namely $\sum_{i=1}^{n-1} (-1)^{n+i} (i-1)^m \binom{n-1}{i-1} = (n-1)! S(m, n-1) - (n-1)^m = (n-1)! S(m, n-1) - \sum_{k=0}^m S(m, k) (n-1)_k < 0$. \square

Remark 4.2. Theorem 4.1 indicates that $4n \log n + cn$ shuffles are enough to mix the deck. This is in line with the results of Aldous and Diaconis (1986) as well as Diaconis et al. (1992).

Acknowledgements

I would like to thank Jason Fulman for helpful discussions and Daniel Panario for his useful comments.

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